On Lie supergroups and superbundles defined via the Baker-Campbell-Hausdorff formula

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Abstract. *The Baker-Campbell-Hausdorff formula is extended to the class of Lie superalgebras and then is used to define a class of objects, which are called Lie supergroups. Such Lie supergroups turn out to be either Lie groups (then the Jacobi $\mathbb{Z}_2$-identity is simultaneously the usual Jacobi identity) or some sets provided with a partially associative multiplication. Thus they are different objects from supergroups in the sense of Kostant, Berezin or A. Rogers. In particular no Grassmann algebra is used in the paper. Examples of*

1) *some matrix Lie superalgebras,*
2) *certain subalgebras of the super-Poincaré algebra*
3) *the special Lie superalgebras $D(2,1,\alpha)$ and their subalgebras*

are presented.

Furthermore, cocycles of functions taking values in the Lie supergroups are defined and their partially associative multiplicative relations are considered. Some fibre bundles, whose transition functions determine such cocycles, are distinguished.

0. INTRODUCTION

The Baker-Campbell-Hausdorff formula, shortly the BCH-formula, is one of the principal and most early discovered relations linking the basic conceptions of linear *algebra*, classical *analysis* and global *geometry*. It was derived in a general form by Baker in 1903-1906 [1, 2] and by Hausdorff in 1906 [11]. This was before the formulation of a general theory of Lie groups and Lie algebras, in particular the so-called fundamental Lie theorem stating the 1-1 correspondence between Lie algebras and connected, simply

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connected Lie groups. It was the BCH-formula which showed a direction of development of this theory.

Many aspects of the BCH-formula, both analytical and geometrical, remain unexplained. Let us observe that the power series of $\frac{z}{1-e^{-z}} - \frac{z}{2}$, which was applied by Hausdorff, appears also in the standard definition of the Todd class [12], so that it is involved in the index theorem. This suggests the existence of many natural relations between cohomologies of Lie groups and coefficients of series given by the BCH-formula. Such coefficients have been analysed recently by Kostelecky et al [15], [16], who have considered the case of Lie supergroups in the sense of Berezin and A. Rogers, i.e. Lie groups admitting an additional Grassmann structure, see [4] and [19]. Let us note that Berezin used the BCH-formula for Lie algebras with a Grassmann structure as early as the late 70's. However his preprint has been developed more fully and translated in English much later, see [3], p. 251.

In this paper the BCH-formula is used in order to define exponents of the Lie superalgebras. We prove that if $\mathfrak{g}$ is a Lie superalgebra and $\exp \mathfrak{g}$ denotes its exponent defined in this way then $\exp \mathfrak{g}$ may be either 1° a Lie group or 2° a local analytic non-associative groupoid in which the left inverse of any element is equal to the right inverse or 3° a local analytic non-associative groupoid admitting different left and right inverses of one element. The said groupoids are associative, however, in the sense of 1- and 2-order terms.

The cases of 1°, 2° and 3° are determined by simple *algebraic* relations in Lie superalgebras. In particular, the first case corresponds to a class of Lie superalgebras whose $\mathbb{Z}_2$-Jacobi identity, becomes the usual Jacobi identity. The suggestion that the Jacobi identity is a key point for the BCH-formula may be found in the original paper of Hausdorff [11].

Among *analytical* problems related to the BCH-formula for Lie superalgebras we discuss the problem of a compatibility of relevant series with the exponential map.

In the final part we apply the Lie supergroups here considered in a global *geometry*, using them as generalized structural groups of some fibre bundles, which are called superbundles. Those superbundles are determined by cocycles of transition functions taking values in Lie supergroups. For those cocycles a partial associativity is sufficient.

In mathematical literature devoted to supergeometry the word «Lie supergroup» or «supergroup» usually corresponds either to a supergroup in the sense of Berezin and A. Rogers or to an object defined by Kostant [14], cf. also [3] and [17]. If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra then its supergroup in the sense of Kostant is a sheaf of Grassmann algebras over the group manifold $\exp \mathfrak{g}$ together with a distinguished algebra of sections of this sheaf. The algebra in question is defined by means of the Hopf algebra methods. For such a structure commuting diagrams corresponding to group axioms are well defined.

A notion which seems to be the purpose of these and related studies may be called a *global Lie supergroup*. In this term the word «global» has its usual meaning, the word «Lie» reflects an almost 1-1 correspondence between Lie superalgebras and the above mentioned Lie supergroups and the word «super-» should refer to an object having a structure more rich and more general but similar to the structure of a group (the «difference» between Lie supergroups and Lie groups should not be «larger» than the «difference» between Lie superalgebras and Lie algebras). This a terminology allows us to say that:

- Lie supergroups of Kostant are «global» and «Lie» but not «super-»;
- Lie supergroups of Berezin are «global» and «super-» but not «Lie»;
- Lie supergroups of this paper are «Lie» and «super-» but not «global».

An effective construction of a global Lie supergroup would be a crowning and realy super achievement in the domain related to Lie superalgebras.

1. LIE SUPERALGEBRAS

DEFINITION. We call *a Lie superalgebra* (eventually simply algebra) a pair $\mathfrak{g} = \langle \mathfrak{g}, (\cdot,\cdot)\rangle$, where $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is $\mathbb{Z}_2$-graded linear space real or complex ($\mathbb{Z}_2 = \langle\{0,1\}, \oplus\rangle$) and $(\cdot,\cdot)$ is a bilinear $\mathfrak{g}$-valued form such that

$$(1) \qquad (x_i, y_j) = -(-1)^{ij}(y_j, x_i) \in \mathfrak{g}_{i\oplus j}, \qquad x_i \in \mathfrak{g}_i, y_j \in \mathfrak{g}_j$$

and the Jacobi $\mathbb{Z}_2$-identity is satisfied that is

$$(2) \qquad (x_i,(y_j,z)) - ((x_i,y_j),z) - (-1)^{ij}(y_j,(x_i,z)) = 0.$$

∎

If for all $x_1, y_1 \in \mathfrak{g}_1$ it is $(x_1, y_1) = 0$ then $\mathfrak{g}$ is a Lie algebra and a Lie superalgebra simultaneously. In this case we will call $\mathfrak{g}$ a *$\mathbb{Z}_2$-graded Lie algebra* if $\mathfrak{g}_1 \neq 0$ and *an even Lie algebra* if $\mathfrak{g}_1 = 0$.

The subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$, which is an even Lie algebra, will be called *the even part* of $\mathfrak{g}$.

A $\mathbb{Z}_2$-graded Lie algebra, whose $\mathbb{Z}_2$-graded vector space is the same as that of Lie superalgebra $\mathfrak{g}$ and whose bracket form is given by

$$(3) \qquad [x,y]^{\sim} := (x,y) - (x_1,y_1),$$

will be called *the Lie part of $\mathfrak{g}$* and denoted $\tilde{\mathfrak{g}}$.

One can quickly derive from (1) and (2) that

$$(4)\qquad \begin{aligned}(x_0, x_0) &= 0,\\ (x_1,(x_1,x_1)) = ((x_1,x_1),x_1) &= 0.\end{aligned}$$

From now on only finite dimensional Lie superalgebras will be considered.

As a fundamental example regard the linear Lie superalgebra $g = gl(p,q)$, whose parts g_0, g_1 consist of matrices of the type resp.

$$(5)\qquad \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}, \; a = a_{p\times p}, b = b_{q\times q}, c = c_{p\times q}, d = d_{q\times p}$$

and whose bracket of elements $x_i \in g_i$, $y_j \in g_j$ is

$$(6)\qquad (x_i, y_j) := x_i y_j - (-1)^{ij} y_j x_i.$$

The generalized Ado theorem says that each real or complex finite dimensional Lie superalgebra is isomorphic to a subalgebra of a linear Lie superalgebra $gl(p,q)$, cf. [13].

2. THE CLASSICAL BAKER-CAMPBELL-HAUSDORFF FORMULA

Let $h = \langle h, [\cdot,\cdot]\rangle$ be a Lie algebra, $x, y \in h$ and t be a real positive number. Put

$$(7)\qquad C(tx, ty) := \sum_{n=1}^{\infty} c_n(x,y)t^n,$$

where coefficients c_n are defined inductively as follows

$$(8)\qquad c_1(x,y) := x + y,$$

$$(9)\qquad \begin{aligned}(n+1)c_{n+1}(x,y) &:= \frac{1}{2}\cdot\left[x - y, c_n(x,y)\right]\\ &+ \sum_{m\geq 1, 2m\leq n} a_{2m} \sum_{\substack{k_1,\dots,k_{2m}>0\\ k_1+\dots+k_{2m}=n}} [c_{k_1}(x,y),[\dots[c_{k_{2m}}(x,y), x+y]\dots],\end{aligned}$$

in which a_{2m} are given by the following formula

$$1 + \sum_{m=1}^{\infty} a_{2m} z^{2m} = \frac{z}{1-e^{-z}} - \frac{z}{2} = 1 + \frac{z^2}{12} - \frac{z^4}{720} + \dots$$

so that

$$a_{2m} = (-1)^{m-1} \frac{B_m}{(2m)!} = (-1)^{m-1} \frac{2}{(2\pi)^{2m}} \sum_{k=1}^{\infty} \frac{1}{k^{2m}}, \quad m = 1, 2, \dots$$

Recall that B_m are Bernoulli numbers. They are rational. The above series (7) is called *the Hausdorff series.*

This definition is close to the original formula of Hausdorff [10], cf. [22]. However, the series (7) may be defined by means of others formulae. Among them most popular is the following direct formula due to Dynkin [8]:

$$C(x,y) :=$$

$$(10) \quad := \sum_{p=1}^{\infty} \frac{(-1)^p}{p} \sum_{\substack{i_1, j_1, \dots, i_p, j_p \geq 0 \\ i_1 + j_1 \geq 1, \dots, i_p + j_p \geq 1}} \frac{1}{i_1 + j_1 + \dots + i_p + j_p} \frac{\left[x^{i_1}, y^{j_1}, \dots, x^{i_p}, y^{j_p}\right]}{i_1! j_1! \dots i_p! j_p!}$$

where

$$\left[x^{i_1}, y^{j_1}, \dots, x^{i_p}, y^{j_p}\right] :=$$

$$:= \underbrace{[x, [x, \dots [x,}_{i_1} \underbrace{[y, \dots [y,}_{j_1} [x, \dots, y, \underbrace{[x, \dots [x,}_{i_p} \underbrace{[y, \dots, y] \dots]}_{j_p}$$

and one can assume that:

$$(11) \quad (i_p = 0 \text{ or } 1) \text{ and } ((j_p = 0) \Rightarrow (i_p = 1)).$$

The direct formulae for $c_2 - c_5$ are as follows

$$(12) \quad c_2(x,y) = \frac{1}{2}[x,y],$$

$$(13) \quad c_3(x,y) = \frac{1}{12}[x,[x,y]] - \frac{1}{12}[y,[x,y]],$$

$$(14) \quad c_4(x,y) = -\frac{1}{24}[x,[y,[x,y]]],$$

$$(15) \quad \begin{aligned} c_5(x,y) = &-\frac{1}{720}[x,[x,[x,[x,y]]]] - \frac{1}{120}[x,[x,[y,[x,y]]]] \\ &+ \frac{1}{360}[y,[x,[x,[x,y]]]] - \frac{1}{360}[x,[y,[y,[x,y]]]] \\ &+ \frac{1}{120}[y,[x,[y,[x,y]]]] + \frac{1}{720}[y,[y,[y,[x,y]]]]. \end{aligned}$$

Consider the Lie algebra polynomials $p_n(x,y) = \sum_{i=1}^{l_n} \rho_{i,n} r_{i,n}(x,y)$ of which elements $r_{i,n}$ are defined inductively

$$
(16) \qquad \begin{gathered} r_{1,1} := x, \quad r_{2,1} := y, \quad r_{1,2} := [x,y], \ldots \\ r_{i,n+1} := \begin{cases} [x, r_{i,n}], & i \le l_n \\ [y, r_{i-l_n,n}] & i > l_n. \end{cases} \end{gathered}
$$

The elements $r_{i,n}$ correspond to homogeneous functions of n-th degree which are determined by such formulae composed of x, y and square brackets whose part at the end is $\ldots x, y\underbrace{]\ \ldots]}_{n-1}$.

From now on the elements $r_{i,n}$ will be understood not in the sense of corresponding functions of x, y on a Lie algebra but they will be identifield with the formulae in (16). Such an identification is helpful because the Jacobi identity makes some elements of type $r_{i,n}$ and $\dot{r}_{j,n}$ identical in sense of functions of x, y on each Lie algebra. In particular we have

$$
\begin{aligned}
[y,[x,[x,y]]] &= [x,[y,[x,y]]], \\
[x,[y,[x,[x,y]]]] &= [x,[x,[y,[x,y]]]], \\
[y,[y,[x,[x,y]]]] &= [y,[x,[y,[x,y]]]],
\end{aligned}
$$

The sets of elements $r_{j,n}$ which determine the same function of x, y as $r_{i,n}$ on each Lie algebra will be denoted by $\{r_{i,n}\}$.

Let us order the set of all elements $r_{i,n}$ in the following way: Assign to $r_{i,n}$ natural numbers whose 0-1 expansions are equivalent to 0-1 sequences associated to sequences of variables x and y which successively occur in $r_{i,n}$ by the correspondence $x \to 1, y \to 0$. Then in each set $\{r_{i,n}\}$ we can distinguish a maximal element $\dot{r}_{i,n}$.

There is the only formula for the Hausdorff series such that the coefficients c_n are combinations of the maximal elements $\dot{r}_{i,n}$. It will be called *the standard formula for the Hausdorff series.* The coefficients $c_2 - c_5$ in (12-15) are written according to this formula.

The problem of a choice of a particular formula for the Hausdorff series in Lie algebra theory is rather marginal. However it turns out to be essential in a case of Lie superalgebras.

If the Lie algebra $\hbar$ is nilpotent then the Hausdorff series is a polynomial so that it is well defined for all $x, y \in \hbar$. In other cases domains of the Hausdorff series should be defined. Therefore consider a norm $|\cdot|$ in $\hbar$ and let $M \ge 1$ be a number such that

$$
(17) \qquad |[x,y]| \le M|x|\,|y|, \quad x, y \in \hbar.
$$

The domain where the series (7) with $t = 1$ is absolutely convergent contains the product $B^{\hbar} \times B^{\hbar}$ of balls

$$B^{\hbar} := B\left(0, \frac{\ln 2}{2M}\right) = \left\{x \in \hbar : |x| < \frac{\ln 2}{2M}\right\}. \tag{18}$$

For the proof see [22], cf. also [5].

If $\hbar$ is considered as a subalgebra of the linear Lie algebra $gl(p)$ then for $x, y \in \hbar$ we have

$$e^{tx}e^{ty} = e^{C(tx,ty)}, \quad e^{x} := \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x|, |y| < \frac{\ln 2}{4t}.$$

The following identities

$$C(tx, -tx) = 0, \tag{19}$$

$$C(C(tx,ty),tz) = C(tx, C(ty,tz)), x, y, z \in B\left(0, \frac{\ln 3/2}{3Mt}\right). \tag{20}$$

reflect invertibility and associativity of the multiplication in any Lie group whose Lie algebra is isomorphic to $\hbar$.

3. THE BCH-FORMULA FOR LIE SUPERALGEBRAS

As a *formal bracket series of* x, y we understand a sum $f(tx, ty) = \sum_{n=1}^{l_n} a_n(x, y)t^n$, whose coefficients $a_n(x, y) = \sum_{i=1}^{l_n} \beta_{i,n} b_{i,n}(x, y)$ are linear combinations of elements

$$b_{1,1} := x, \quad b_{2,1} := y, \quad b_{i,n} := (b_{j,p}, b_{k,n-p}),$$
$$\text{where } p = 1, \ldots, n-1, \quad i = \varphi(j, k, p)$$

and $\varphi : \mathbb{N}^3 \to \mathbb{N}$ is an injective function. The variables x, y will be considered as elements of a Lie superalgebra $g = g_0 \oplus g_1$ and then the brackets of f will be treated as the brackets of g.

If for each Lie superalgebra $g = g_0 \oplus g_1$ the restriction of $f(x, y)$ to g_0 is equal to the Hausdorff series on g_0 (in the sense of functions of x, y on B^{g_0}) then $f(x, y)$ will be called *the superalgebraic Hausdorff series.*

We will deal usually with superalgebraic Hausdorff series whose inductive procedure does not admit elements $(x, x), (y, y)$ and $(b_{i,m}, b_{j,n})$ with $m, n > 1$ and assume that such series admit as a low estimate of convergency radius a simply generalization of the estimate (18) of convergency radius of the Hausdorff series. Let us define these series explicitely

DEFINITION. A superalgebraic Hausdorff series $f(tx, ty)$ is called *the BCH-product* if

B1. The elements $b_{i,n}$ which admit non-zero coefficients are defined inductively

$$b_{1,1} := x, b_{2,1} := y, \quad b_{1,2} := [x, y], \quad b_{2,2} := [y, x],$$
$$b_{4i,n+1} := [x, b_{i,n}], \quad b_{4i+1,n+1} := [y, b_{i,n}],$$
$$b_{4i+2,n+1} := [b_{i,n}, x], \quad b_{4i+3,n+1} := [b_{i,n}, y].$$

B2. Let $|\cdot|$ be a norm in a Lie superalgebra g and M be a real number such that

(21) $$|(x,y)| \leq M|x||y| \quad \text{for } x, y \in g \text{ and } M \geq 1.$$

Then series $\sum_{n=1}^{\infty} a_n(x,y)$ is absolutely convergent for $x, y \in B^g$, where

(22) $$B^g := \left\{ x \in g : |x| < \frac{\ln 2}{2M} \right\}.$$

■

For the BCH-products we will use notations

(23) $$\begin{aligned} D(tx, ty) = D_\mu(tx, ty) &= \sum_{n=1}^{\infty} d_n(x,y) t^n \\ &= (x+y)t + \left\{ \frac{\mu}{2}(x,y) - \frac{1-\mu}{2}(y,x) \right\} t^2 + \dots \end{aligned}$$

BCH-products $D_{1/2}$ corresponding to $\mu = \frac{1}{2}$ will be called *Jacobi products.* A relation between them and the Jacobi identity will be discussed in the next section. The Jacobi products will be denoted $D^{(J)} = \sum d_n^{(J)} t^n$. Then we have

$$d_2^{(J)} = \frac{1}{4}(x,y) - \frac{1}{4}(y,x) = \frac{1}{2}(x_0, y_0).$$

If we put in the series directly defined by the recursive formula (9) the round brackets instead of the square ones then we do not obtain any BCH-product. The coefficients of t^2 in the resulting superalgebraic Hausdorff series is

$$\frac{1}{4}(x,x) + \frac{1}{4}(x,y) - \frac{1}{4}(y,x) - \frac{1}{4}(y,y).$$

The following series are examples of BCH-products.

a) The *standard BCH-products* $D^{(s)}$. It may be obtained by «rounding» all square brackets in the standard formula for the Hausdorff series from the previous section. The expansion of $D^{(s)}$ up to t^3 looks as follows

$$
\begin{aligned}
D^{(s)}(tx,ty) &= \sum_{n=1}^{\infty} d_n^{(s)}(x,y)t^n = \{x+y\}t + \frac{1}{2}(x,y)t^2 \\
&+ \frac{1}{12}\{(x,(x,y)) - (y,(x,y))\}t^3 + \dots
\end{aligned} \tag{24}
$$

The below series are Jacobi products.

b) The *balanced BCH-product* $D^{(b)}$. It may be formed by replacing in the standard BCH-product (x,y) with $\frac{1}{2}(x,y) - \frac{1}{2}(y,x)$ everywhere so that

$$
\begin{aligned}
D^{(b)}(tx,ty) &= \{x+y\}t + \frac{1}{4}\{(x,y) - (y,x)\}t^2 + \\
&+ \frac{1}{24}\{(x,(x,y)) - (x,(y,x)) - (y,(x,y)) \\
&+(y,(y,x))\}t^3 + \dots
\end{aligned} \tag{25}
$$

This product satisfies the equality of invertibility (19).

c) The *trivial BCH-product* $D^{(tr)}$. It is obtained when replacing of (a,b) with $\frac{1}{2}(a,b) - \frac{1}{2}(b,a)$ is carried out for each pair of brackets in the formula defining the standard BCH-product. Thus in particular

$$
\begin{aligned}
\frac{48}{t^3}d_3^{(tr)}(x,y) &= (x,(x,y)) - (x,(y,x)) \\
&- (y,(x,y)) + (y,(y,x)) - ((x,y),x) + ((y,x),x) \\
&+ ((x,y),y) - ((y,x),y).
\end{aligned} \tag{26}
$$

Let $\tilde{C}_g$ denote the Hausdorff series for the Lie part $\tilde{g}$ of a Lie superalgebra g, cf. (3). One can quickly check that

$$
D^{(tr)}(tx,ty) = \tilde{C}_g(tx,ty) \quad \text{for} \quad tx,ty \in B^g. \tag{27}
$$

Thus the trivial BCH-product may be interpreted as a superalgebraic formula for the Hausdorff series in $\mathbb{Z}_2$-graded Lie algebras. This fact explains the name «trivial».

d) The *Dynkin BCH-product* $D^{(D)}$. It comes from Dynkin's formula so that

$$
\begin{aligned}
D^{(D)}(tx,ty) &= \{x+y\}t + \frac{1}{4}\{(x,y) - (y,x)\}t^2 + \frac{1}{36}\{(x,(x,y)) \\
&-2(x,(y,x)) + (y,(y,x)) - 2(y,(x,y))\}t^3 \\
&+ \frac{1}{48}\{(y,(x,(y,x))) - (x,(y,(x,y)))\}t^4.
\end{aligned} \tag{28}
$$

Let us distinguish three classes of Lie superalgebras for which a considerable part of elements of BCH-products vanish.

1. Consider a Lie superalgebra g such that

$$(g_0, g_0) = (g_0, g_1) = 0 \tag{29}$$

so that the bracket of g is a symmetric form. In this case g will be called *a BCH-Abelian Lie superalgebra.* For algebras of this type BCH-products $D = D_\mu$ are the sum of a linear and a quadratic form of type

$$D_\mu(tx, ty) = (x + y)t + (\mu - \frac{1}{2})(x_1, y_1)t^2. \tag{30}$$

2. The next class of Lie superalgebras is determined by

$$(g_1, (g, g_1)) = 0. \tag{31}$$

The $\mathbb{Z}_2$-Jacobi identity is simultaneously the usual Jacobi identity if and only if the above equality is satisfied because then $(y_1, (x_1, z)) = 0$, cf. (2).

Le us recall the original paper of Hausdorff [10], where a general algebraic formulation of the BCH-formula was elaborated. In this paper only anti-symmetric bracket forms were considered. Nevertheless, the author made of the Jacobi identity the main postulate and his comment may be interpreted in this way that from a viewpoint of the BCH-formula the Jacobi identity is the fundamental axiom of Lie algebras.

We prove as soon as that if g satisfies (31) then BCH-products determine the equality of associativity like (20) which is due to Lie algebras and the classical BCH-formula. That is why if (31) is fulfilled we will call g *a BCH-Lie superalgebra.*

Let g be a BCH-Lie superalgebra, $D = D_\mu$ be a BCH-product and $D^{(J)}$ be a Jacobi product. Then we have

$$D(tx, ty) = \tilde{C}_g(tx, ty) + (\mu - \frac{1}{2})(x_1, y_1)t^2, \tag{32}$$

$$D^{(J)}(tx, ty) = D^{(tr)}(tx, ty) = \tilde{C}_g(tx, ty), \tag{33}$$

cf. (30).

3. In this point we distinguish such Lie superalgebras that

$$(g_0, (g_1, g_1)) = 0. \tag{34}$$

This condition is equivalent to

$$(x, (x, x)) = ((x, x), x) = 0 \quad \text{for each } x \in g. \tag{35}$$

If the above conditions are satisfied then g will be called *a BCH-invertible Lie superalgebra.* Such a name is a consequence of the third point of the below theorem. Moreover, Lie superalgebras which are BCH-invertible and are not BCH-Lie superalgebras will be called *weak BCH.*

THEOREM.

1. Let D be a BCH-product. The identity

$$D(tx,ty) = D(ty,tx), \quad x,y \in g \tag{36}$$

takes place if and only if g is a BCH-Abelian Lie superalgebra.

2. If g is a BCH-Lie superalgebra and $D = \sum d_n t^n$ is a BCH-product then for $x, y, z \in B\left(0, \frac{\ln 3/2}{3Mt}\right)$, cf. (20) and (21), it is

$$D(D(tx,ty),tz) = D(tx,D(ty,tz)). \tag{37}$$

If $d_2 \neq d_2^{(tr)}$ or $d_3 \neq d_3^{(tr)}$, cf. (25), then the above equality is satisfied for all $x, y, z \in B\left(0, \dfrac{\ln 3/2}{3Mt}\right)$ if and only if g is a BCH-Lie superalgebra.

3. If g is a BCH-invertible Lie superalgebra and D is a BCH-product, then for each $x \in g$ there exists $y \in g$ such that

$$D(x,y) = D(y,x) = 0. \tag{38}$$

If g is a Lie superalgebra and for each $x \in g$ there exists $y \in g$ such that

$$D^{(s)}(x,y) = D^{(s)}(y,x) = 0 \tag{39}$$

then g is BCH-invertible.

In case of $D = D^{(b)}$, see (25), and of any Lie superalgebra g it is

$$D^{(b)}(x,-x) = D^{(b)}(-x,x) = 0 \tag{40}$$

for each $x \in g$.

The proof is in [9].

Comments. (c) In the third point $D^{(s)}$ and $D^{(b)}$ determine the extreme cases. In remaining cases conditions of BCH-invertibility are non-trivial but may be weaker than that of $D^{(s)}$. An example is given by $D = D^{(D)}$, see (27), when it is

$$y_1 = z_1 = -x, \quad y_i = z_i = 0 \quad \text{for} \quad i = 2,3,4$$
$$\text{and} \quad y_5 = -z_5 = \frac{1}{60}(x,(x,(x,(x,x)))).$$

(cc) If D in (37) will be replaced by a Hausdorff series admitting terms $\kappa(x,x)t^2$ or $\lambda(y,y)t^2$ then there will be usually not any identity between t^2-order terms in L and P.

One can expect that BCH-Lie superalgebras correspond to Lie groups and BCH-invertible Lie superalgebras correspond to some locally symmetric spaces.

STATEMENT. 1° Let $\mathfrak{g}$ be a BCH-Lie superalgebra, D be a BCH-product and put $x \circ y := D(x,y)$, where $x, y \in B^{\mathfrak{g}}$. Then there exists a unique connected, simply connected Lie group G which is locally isomorphic to $\langle B^{\mathfrak{g}}, \circ \rangle$. Such groups corresponding to different BCH-products and isomorphic Lie superalgebras are isomorphic. The group G is abelian iff $\mathfrak{g}$ is BCH-Abelian.

2° Let $\mathfrak{g}$ be a BCH-invertible Lie superalgebra. Then for every $x \in B^{\mathfrak{g}}$ there is a curve $c_t \subset B^{\mathfrak{g}}$ through 0 and x such that $\langle c_t, \circ \rangle$ is a local 1-parameter group.

Proof. 1° If $\mathfrak{g}$ fulfils (31) then the following limits are well defined for each $x, y \in B^{\mathfrak{g}}$.

$$(x+y) := \lim_{n\to\infty} \left(l_x\left(\frac{1}{n}\right) \cdot l_y\left(\frac{1}{n}\right) \right)^n ,$$

$$[x,y] := \lim_{n\to\infty} \left(l_x\left(\frac{1}{n}\right) \cdot l_y\left(\frac{1}{n}\right) \cdot l_x\left(-\frac{1}{n}\right) \cdot l_y\left(-\frac{1}{n}\right) \right)^{n^2}$$

and they determine a Lie algebra structure $\hat{\mathfrak{g}}$ at the space of $\mathfrak{g}$.

By a direct calculation one can check that $\hat{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}$ given in (3) are isomorphic. Thus in particular if $\mathfrak{g}$ is BCH-Ábelian then $\hat{\mathfrak{g}}$ is abelian. The desired Lie group is that which is assigned to $\hat{\mathfrak{g}}$ according to the fundamental Lie theorem.

2° Let us trace a local trajectory $c_t = \{l_x t\}$ through 0 and any point $x = x_0 + x_1 \in B^{\mathfrak{g}}$ by the formula

$$l_x t := x_0 t + (\mu - \frac{1}{2})(x_1, x_1)t^2 + x_1 t, \quad x \in [-1, 1] \tag{41}$$

where μ and $D = D_\mu$ are as in (23). Then we have

$$l_x t \circ l_x u = l_x(t+u) + a_3(t,u)((x_1,x_1),x_0) + b_3(t,u)(x_0,(x_1,x_1)) + \dots$$

Hence there are two main cases where $\langle c_t, \circ \rangle$ is a local group: 1° $\mathfrak{g}$ is BCH-invertible, 2° $D = D^{(b)}$. The first case determines the proving statement. ∎

EXAMPLE. Let $x_1 \in \mathfrak{g}_1$ be such that $(x_1, x_1) \neq 0$. Then we can assign to x_1 a 2-dimensional BCH-abelian subalgebra $\mathfrak{g}^{(x_1)} \subset \mathfrak{g}$ such that $\mathfrak{g}_0^{(x_1)} := \{(x_1, x_1)\}$, $\mathfrak{g}_1^{(x_1)} := \{x_1\}$. For $y_i = a_i(x_1, x_1) + b_i x_1$, $i = 1, 2$, we have

$$y_1 \circ y_2 = \left(a_1 + a_2 + \left(\mu - \frac{1}{2} \right) b_1 b_2 \right) (x_1, x_1) + (b_1 + b_2) x_1 . \tag{42}$$

The mapping $a(x_1, x_2) + b x_1 \to \langle a - \frac{2\mu - 1}{4} b^2, b \rangle$ is an isomorphism between $\mathfrak{g}$ and the additive group $\mathbb{R}^2$ or $\mathbb{C}^2$.

REMARKS. (r) The local trajectories c_t (41) may be considered as integral lines through 0 of «invariant super-vector fields». However, in the case where g is not BCH-invertible the «full integral» of such a field is a «smeared line», more exactly it is a sequence of fan-like surfaces given by

$$l_x t;\ l_x t_1 \circ l_x t_2;\ (l_x t_1 \circ l_x t_2) \circ l_x t_3,\ l_x t_1 \circ (l_x t_2 \circ l_x t_3); \ldots \tag{43}$$

The dimension of that surface which corresponds to $t_1, \ldots, t_k$ is $\leq \min(k, \dim_{\mathbf{R}} g)$.

The above sequences of surfaces may be considered as representatives of the corresponding invariant super-vector fields. Thus, in this approach, super-vector fields are objects of an *intrinsic* geometry.

A vector calculus based on the above notion of super-vector fields would be an alternative for the standard calculus of supervectors and supertensors due to Kostant [14]. In their theory supervectors are defined as operators in Grassman sheaves over manifolds supporting such vectors so that they are rather objects of an *extrinsic* geometry.

(rr) We can generalize coefficients d_n in case $n = 3, 4, \ldots$ assuming them to be g-forms on $g \times g$ of n-th degree which on $g_0 \times g_0$ coincide with the coefficients c_n. Then we could claim for such generalized BCH-products better algebraic properties, e.g. associativity for a larger class of Lie superalgebras. In this way we can define such series which are associative up to 3rd degree since in the equality of associativity for coefficients by t^3 there is $\frac{n^3(n+1)}{2} - \frac{p^3(p+1)}{2}$, variables more than equations, where $n = \dim g$, $p = \dim g_0$.

(rrr) Recall that the Hausdorff series admits the analytic continuation from $B^{h} \times B^{h}$ on the product $h \times h$ of each Lie algebra h. The problem is what is a class of BCH-products admitting the analytic continuation from $B^{g} \times B^{g}$ on $g \times g$ for each of some Lie superalgebra g. Observe that if g satisfies

$$(g_1, (g, (g, \ldots, (g, g_1) \ldots) = 0$$

then for every BCH-product D there exists such a continuation because then $D - \tilde{C}_g$ is a polynomial.

4. A DISCUSSION ABOUT LIE SUPERGROUPS DEFINED VIA THE BCH-FORMULA

Our intention is to introduce as Lie supergroups corresponding to Lie superalgebras g certain groupoids, i.e. pairs $\langle S, * \rangle$ in which S is a set and « $*$ » is a two-elements product, such that s1. The product « $*$ » is determined by a BCH-product.

s2. The product « $*$ » is well-defined for each pair of points belonging to the image of B^{g} by an exponent-like function.

s3. The analytic continuation of « $*$ » in directions at g_0 and at some special subalgebras of g like BCH-Abelian or BCH-ones may be done.

DEFINITION. Let g be a matrix Lie superalgebra and $A \subset g$, $C \subset g \times g$ denote such sets that

$$B^{g} \times B^{g} \subset C \subset A \times A.$$

Furthermore, D is a BCH-product admitting analytic continuation (possibly multivalued) $\bar{D}$ from $B^{g} \times B^{g}$ on C such that

$$\bar{D}(C) \subset A.$$

The pair $\langle \mathcal{G}, * \rangle$ is called *a Lie-supergroup* (in the sense of the BCH-formula) if $\mathcal{G} = e^{A}$ is the image of the set A by e^{x} and « $*$ » is a map from $(e \times e)(C)$ in $\mathcal{G}$ given by

$$e^{x} * e^{y} := e^{\bar{D}(x,y)}, \quad \text{for} \quad \langle x, y \rangle \in C. \tag{44}$$

According to algebraic terminology [6] the Lie supergroups are some local (partial) groupoids with one unit element. The Lie supergroups which correspond to BCH-invertible Lie superalgebras are local quasi-groups (i.e. left inverse elements are equal locally to right inverse ones). In the case of BCH-Lie superalgebras the Lie supergroups become Lie groups.

Lie supergroups are isomorphic if the corresponding local groupoids are isomorphic and the isomorphism is an analytical map. An isomorphism between Lie supergroups $\mathcal{G}$ and $\mathcal{G}'$ corresponding to Lie superalgebras g and g' resp. and to a common BCH-product D is determined by an analytical 1-1 map $f : A \to A'$ such that

1. $\langle f(x), f(y) \rangle \in C'$ iff $\langle x, y \rangle \in C$,
2. $f(\bar{D}(x,y)) = \bar{D}(f,(x), f(y))$, $\quad \langle x, y \rangle \in C$.

Let g be a BCH-Lie algebra and $\tilde{g}$ be the Lie part of g. Then the map

$$f(x) := x + \frac{1}{4}(x_1, x_1) \tag{45}$$

determines an isomorphism between Lie supergroups $\mathcal{G}$ and $\tilde{\mathcal{G}}$ corresponding to g and $\tilde{g}$ resp. and to $D = D^{(s)}$ if the corresponding domains of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are compatible with f.

In order to have more simple and effective relations between Lie superalgebras and Lie supergroups we describe below in points $a - e$ five particular cases and variants of the above definition.

a) Let us put in the definition

$$D := D^{(s)}, \quad A := \mathcal{g}, \quad C := B\mathcal{g} \times B\mathcal{g}. \tag{46}$$

The resulting Lie supergroups will be called *the standard ones.*

The isomorphism between standard Lie supergroups $\mathcal{G}_s$ and $\mathcal{G}'_s$ may be determined only by such analytic maps from $\mathcal{g}$ in $\mathcal{g}'$ which satisfy $f(B\mathcal{g}) = B\mathcal{g}'$. That is why the map f given by (45) does not determine any isomorphism between the standard Lie supergroups corresponding to $\mathcal{g}$ and $\tilde{\mathcal{g}}$.

The problem is whether the correspondence between Lie superalgebras and standard Lie supergroups up to isomorphisms is 1-1 or not. In the special case where only linear isomorphisms of standard Lie supergroups are taken into account (i.e. maps f are linear) the analogous correspondence is 1-1. This is a consequence of

$$x + y = \frac{d}{dt}D(tx, ty)\big|_{t=0}, \qquad (x, y) = \frac{d^2}{dt^2}D^{(s)}(tx, ty)\big|_{t=0}.$$

The domain $C = B\mathcal{g} \times B\mathcal{g}$ in (46) is not maximal. One of its admissible extensions is $C \cup \mathcal{g}_0 \times \mathcal{g}_0$. In the case of a complex subalgebra of $\mathcal{gl}(p, q; \mathbb{C})$ with unit matrix $I = I_{p+q}$ an extension is given by translations of C along vectors $\langle 2\pi i m I,\ 2\pi i n I\rangle, m, n \in \mathbb{Z}$, because

$$e^{D(x+2\pi i m I, y+2\pi i n I)} = e^{D(x,y)}$$

In contrast to the case of Lie algebras there is an inconsistencey between e^x and non-trivial BCH-products, which obstruct the enlarging of a domain of « $*$ » from $(e \times e)(B\mathcal{g} \times B\mathcal{g})$ to $(e \times e)(\mathcal{g} \times \mathcal{g})$. In order to see it we put

$$x^1 = y^1 := 0, \qquad x^2 := \begin{bmatrix} 0 & 2\pi \\ -2\pi & 0 \end{bmatrix}, y^2 := \begin{bmatrix} 0 & -2\pi \\ 2\pi & 0 \end{bmatrix}.$$

One can calculate

$$e^{x^{1,2}} = e^{y^{1,2}} = e^{D^{(s)}(x^1, y^1)} = I_2,$$
$$\text{but} \quad e^{D^{(s)}(x^2, y^2)} = e^{4\pi^2} I_2 \neq I_2.$$

The above inconsistency between e^x and $D^{(s)}$ does not take place if all elements of $\mathcal{g}$ are nilpotent matrices. This fact follows from the superalgebraic Engel theorem, see

[13] and a simple matrix calculation. In this case formula (44) is valid for $A \times A = C = g \times g$ so that the corresponding standard Lie supergroups extend to groupoids of type $(e^{g}, *)$.

b) In the above example the inconsistency between e^x and $D^{(s)}$ takes place for elements of the BCH-Abelian subalgebra of $gl(1,1)$ generated by x^2 and y^2. However, for Jacobi products $D^{(J)}$ and BCH-Lie superalgebras f there is no inconsistency of this type. This is so because $D^{(J)}$ on f is equal to the Hausdorff series corresponding to the Lie part $\tilde{f}$ of f, cf. (27). Thus the following family of Lie supergroups is well defined

$$D := D^{(J)},\ A := g,\ C := C_J := B^{g} \times B^{g} \cup \bigcup_{f \in S(g)} f \times f, \tag{47}$$

where $f \in S(g)$ iff f is a subalgebra of g and $(f_1, (f, f_1)) = 0$.

The Lie supergroups defined in this item will be called *the Jacobi-Lie supergroups* (recall that the subalgebras $f \in S(g)$ admit the classical Jacobi identity). The set $S(g)$ will be called the *spectrum of BCH-subalgebras of* g.

Note that a 1-dimensional subalgebra $\{x\} \subseteq g$ belongs to $S(g)$ iff $(x,x) = 0$. If g is BCH-invertible then each 2-dimensional non-trivial and irreducible Lie superalgebra $f \subset g$ is BCH-Abelian so that $f \in S(g)$. In this case f admits as generators x and (x,x), where $(x,x) \neq 0$.

In contrast to the case of standard Lie supergroups the correspondence between Lie superalgebras and Jacobi-Lie supergroups up to linear isomorphisms is not one to one. An example of a pair of non-isomorphic Lie superalgebras admitting the same Jacobi-Lie supergroup can be seen in any BCH-Lie superalgebra g and its Lie part $\tilde{g}$.

In the general case Jacobi-Lie supergroups are identical up to a linear isomorphism iff the following identities are satisfied:

$$d_n(x,y) = d'_n(x,y), \quad \text{for all} \quad x, y \in B^{g} \text{ and } n = 2,3,4,5, \tag{48}$$

(then $d_n = d'$ for $n \in \mathbb{N}$).

The equality $d_2 = d'_2$ implies the decomposition of $(\cdot,\cdot)'$

$$(x,y)' = (x,y) + /x_1, y_1/, \quad \text{where} \quad /x_1, y_1/ = /y_1, x_1/, \tag{49}$$

The equalities $d_n = d'_n$, $n \in \mathbb{N}$, follow from the pair of identities

$$(/g_1, g_1/, g_1) = 0, \quad /(g_0, g_1), g_1/ = 0. \tag{50}$$

I do not know any minimal system of equations generating in the general case the equalities $d_n = d'_n$.

We can expect that the above conditions remain unaltered in the case of arbitrary isomorphic Jacobi-Lie supergroups because any isomorphism of such supergroups induces linear isomorphisms of the corresponding BCH-subalgebras.

In the case of Lie superalgebras consisting of nilpotent matrices Jacobi-Lie supergroups admit extension to whole spaces $e^{\mathfrak{g}}$. I do not know other examples of such extensions when $D^{(J)}$ is not trivial on $\mathfrak{g}$.

c) Recall that the parts of Jacobi-Lie supergroups which correspond to subalgebras $f \in S(\mathfrak{g})$ are not arbitrary connected Lie groups having $\tilde{f}$ as Lie algebras but they are Lie groups of type e^{f}. This restriction may be dropped if the definition of Jacobi-Lie supergroups is *generalized* so that e^x in (44) is replaced by a function $E(x)$ defined on C_J such that

C1. $E(x) = e^x$ for $x \in B^{\mathfrak{g}}$

C2. $E(x)_{|f} = \exp_f(x)$, where $\exp_f(\cdot)$ is the exponential map from f in a Lie group K such that $\tilde{f}$ is the Lie algebra of K

C3. $E(x)$ is continuous.

Observe that C3 implies a compatibility of first homotopies of Lie groups corresponding to intersecting subalgebras $f^1, f^2 \in S(\mathfrak{g})$ and the equality of first homotopies of Lie groups corresponding to elements of each continuous family $f^\alpha \in S(\mathfrak{g})$. This suggests that the topology of the resulting *generalized Jacobi-Lie supergroups* is closely connected with the geometry of spectra of BCH-subalgebras.

It follows from this and the previous item that the correspondence between Lie superalgebras and the generalized Jacobi-Lie supergroups up to linear isomorphisms is 1-1 up to discrete Abelian normal subgroups of $\exp f$, $f \in S(\mathfrak{g})$, and symmetric forms $/x_1, y_1/$ admitting decomposition (49) and equalities (48).

The following *modified Lie supergroups* are determined by non-exponential maps of Lie superalgebras which will replace e^x and exponential maps in the definition of Lie supergroups.

d) Consider the simplest case of such a modification when e^x is replaced in (44) by the identity function. Let $D = D_\mu$ be a non-trivial BCH-product with a parameter μ defined in (23). The resulting supermultiplication will be denoted by «o». It satisfies

$$(51) \qquad \begin{aligned} x \circ y &= D_\mu(x, y), \quad x, y \in B^{\mathfrak{g}}, \\ x \circ y &= x_0 + y_0 + \mu(x_1, y_1), \quad x, y \in a, \end{aligned}$$

where $a \subset \mathfrak{g}$ is a BCH-Abelian subalgebra. Thus the following set C_A may be considered as a domain of «o»

$$C_A := B^{\mathfrak{g}} \times B^{\mathfrak{g}} \cup \bigcup_{a \in T(\mathfrak{g})} a \times a,$$

where $T(\mathfrak{g})$ is a set (*spectrum*) of all BCH-Abelian subalgebras of $\mathfrak{g}$.

Let us note that for a given BCH-product D the correspondence between Lie superalgebras and *modified Lie supergroups corresponding to* C_A *and* «o» up to linear isomorphisms is 1-1 if and only if D is not any Jacobi product.

Let us observe that if g is nilpotent then «o» in (51) is well defined for all $x, y \in g$. Furthermore, if all elements of g are nilpotent matrices and both D and a domain C are the same for a « $*$ » – and a «o» – Lie supergroup then both these supergroups are isomorphic (because then e^x is injective on g).

e) Let us observe that the Abelian groups $\langle a, \circ_{|a}\rangle$, which correspond to BCH-Abelian subalgebras of g, are simply connected. This suggests some generalization of the procedure of the previous point similar to that of point (c); such a generalization would allow us to obtain some modified Lie supergroups admitting tori as parts corresponding to $a \in T(g)$. Therefore let us distinguish a subset $\Gamma \subset \bigcup_{a \in T(g)} a$ such that

E1. The intersection $\Gamma \cap a$ is a discrete subgroup of a for each $a \in T(g)$.

E2. $\gamma \circ b \notin B^g$ for each $\gamma \in T$ and $b \in B^g$.

Then the quotient set $C_{A_{/\Gamma}}$ and the quotient of «o» are well defined. The corresponding groupoid will be called the *toroidal Lie supergroup.*

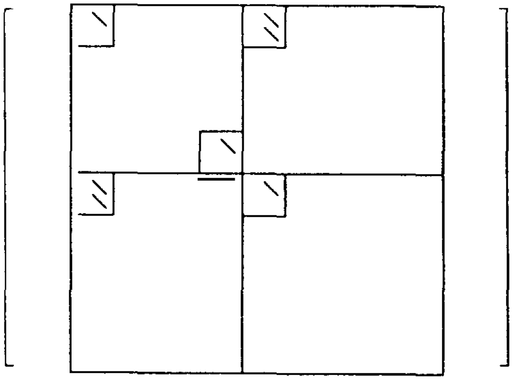
In the above attemps we formed as Lie supergroups some groupoids which are similar to Lie groups in some domains of «almost Lie» subalgebras. The other direction is to look for maximal domains for the analytic continuation of BCH-products.

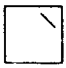
5. EXAMPLES

In this section we will point out BCH- and BCH-invertible subalgebras in matrix Lie superalgebras, special Lie superalgebras of series $D(2,1,\alpha)$ and in the super-Poincaré algebra.

5.1. Matrix Lie superalgebras

a) Recall linear Lie superalgebras $gl(p,q)$, cf. (5-6), and assume that $r|p$ and $r|q$. Then we can distinguish a subalgebra whose any matrix x consists of blocks x_{ij} being $r \times r$ triangular matrices, say upper triangular. This algebra admits subalgebras for which the triangles containing non-zero elements have s elements in first rows if x_{ij} takes place in the g_0-part of x and t elements if it takes place in the g_1-part. Thus the matrix x looks like

(52) $x =$  ,

where  corresponds to $\overbrace{\begin{bmatrix} & a_{ij} \\ 0 & \end{bmatrix}}^{r}\}s$,

and  corresponds to $\overbrace{\begin{bmatrix} & c_{ij} \\ 0 & \end{bmatrix}}^{r}\}t$.

Such subalgebras are well defined if $r > s, r > t$ and $r - 2s \geq t$. We call them *block-triangular Lie superalgebras* and denote $tl(p, q; r; s, t)$.

One can associate with equalities (29), (31), (34) non-empty sets of r, s, t such that the relevant block-triangular subalgebras satisfy these equalities. In particular the following relations

$$2r - 2t - \max(s,t) \geq 0, \quad 2r - s - 2t \geq 0$$

determine such subalgebras which satisfy (31) and (34) resp. The block-triangular Lie superalgebras provide accurate examples for superbundles as we will see in the next secticn.

a) Note that the aforementioned Lie superalgebra $sl(1,1) \subset gl(1,1)$ given by $x_0 = \lambda I_2$ is nilpotent of 2nd order, i.e. $(\mathcal{g}, (\mathcal{g}, \mathcal{g})) = 0$. It is BCH-Abelian and the map

$$x = \begin{bmatrix} a & c \\ d & a \end{bmatrix} \to \langle a - cd, c, d \rangle$$

is an isomorphism between the corresponding Jacobi-Lie supergroup and $\mathbf{R}^3$ resp. $\mathbb{C}^3$.

b) Subalgebras of *orthosymplectic Lie superalgebras* $\mathrm{osp}(p,r)$. Elements of such Lie superalgebras are represented by matrices of type

$$\left[\begin{array}{c|cc} a & \multicolumn{2}{c}{0} \\ \hline 0 & b & c \\ & d & -b^t \end{array}\right] \oplus \left[\begin{array}{c|cc} 0 & x & y \\ \hline -y^t & & \\ & \multicolumn{2}{c}{0} \\ x^t & & \end{array}\right], \quad \begin{array}{l} \text{where} \quad a = a_{p\times p},\ b = b_{r\times r}, \\ c = c_{r\times r},\ d = d_{r\times r} \\ \text{and} \quad a = -a^t,\ c = c^t,\ d = d^t. \end{array}$$

As an example of BCH-Abelian subalgebras when p and r are even let us distinguish the algebra given by

$$a = \begin{bmatrix} \tilde{a} & -\tilde{a} \\ \tilde{a} & -\tilde{a} \end{bmatrix}, \quad b = \begin{bmatrix} \tilde{b} & \tilde{b} \\ -\tilde{b} & -\tilde{b} \end{bmatrix}, \ c = 0,$$

$$d = \begin{bmatrix} \tilde{d} & \tilde{d} \\ \tilde{d} & \tilde{d} \end{bmatrix}, \quad x = \begin{bmatrix} \tilde{x} & \tilde{x} \\ \tilde{x} & \tilde{x} \end{bmatrix}.$$

In order to obtain an example of a non-BCH-Abelian but BCH-subalgebra if $r = 4s$ it suffices to take instead of $c = 0$

$$c = \begin{bmatrix} \tilde{c}_1 & \tilde{c}_2 \\ \tilde{c}_2 & \tilde{c}_3 \end{bmatrix}$$

and to put

$$\tilde{b} = \begin{bmatrix} \underset{\sim}{b} & 0 \\ 0 & \underset{\sim}{b} \end{bmatrix}, \quad \tilde{c}_i = \begin{bmatrix} 0 & \underset{\sim}{c}_i \\ a & c_i \end{bmatrix}, \ i = 1,2,3,$$

$$\tilde{d} = \begin{bmatrix} \underset{\sim}{d} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} \underset{\sim}{x} & 0 \\ 0 & 0 \end{bmatrix}.$$

As an example of weak BCH subalgebras let us point out

$$\left[\begin{array}{c|cc|cc} 0 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} \\ \hline 0 & \tilde{b} & 0 & \multicolumn{2}{c}{0} \\ & 0 & b & & \\ \hline 0 & \tilde{d} & -\tilde{d} & -\tilde{b}^t & 0 \\ & -\tilde{d} & d & 0 & -\tilde{b}^t \end{array}\right] \oplus \left[\begin{array}{cc|cc|c} \multicolumn{2}{c|}{0} & \tilde{x} & -\tilde{x} & 0 \\ & & -\tilde{x} & \tilde{x} & \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 0 \\ \hline \tilde{x}^t & -\tilde{x}^t & \multicolumn{2}{c|}{0} & 0 \\ -\tilde{x}^t & \tilde{x}^t & & & \end{array}\right],$$

where $\tilde{d} = \tilde{d}^t$.

c) Subalgebras of strange Lie superalgebras of serie $P(n), n \geq 3$. Let $P_0(n) = sl(n)$ be as a set and $P_1(n) = gl(n) \oplus a(n)$, where $s(n)$ and $a(n)$ consist of symmetric and skew-symmetric matrices resp. Lie superalgebras $P(n)$ are defined by the following formulae

$$(x_0, x_0') := x_0 x_0' - x_0' x_0, x_0, x_0' \in P_0(n),$$

$$(x, \dot{s}) := xs + sx^t =: -(s, x), \quad s \in s(n), \quad (x, a) := -x^t a - ax =: -(a, x),$$

$$a \in a(n),$$

$$(s, a) := sa =: (a, s), \quad (a, a') = (s, s') = 0.$$

Then in case $n = 2m$ subalgebras whose parts in $P_0(n)$ and $P_0(n)$ may be given by

$$\begin{bmatrix} a & \alpha I_m \\ -\alpha I_m & a \end{bmatrix}, \; a^t = -a, \qquad \begin{bmatrix} \beta I_m & \gamma I_M \\ -\gamma I_m & \beta I_m \end{bmatrix}$$

turn out to be BCH-Lie superalgebras.

5.2. A spectrum of all BCH- and BCH-invertible subalgebras of $D(2, 1, \alpha)$

In order to imagine a Lie supergroup one should see a whole spectrum of BCH- and BCH-invertible subalgebras of the corresponding Lie superalgebra. In the case of a contragredient Lie algebra such a spectrum may be calculated by means of the contragredient decomposition [13]. We will use this method while studying exceptional (*exotic*) continuous series of 17-dimensional Lie superalgebras $D(2, 1, \alpha)$.

The contragredient decomposition of a Lie superalgebra $\mathfrak{g} = D(2, 1, \alpha)$ may be described as follows. Let $\mathfrak{g}^m$, $m \in \mathbb{Z}$, be subspaces of $\mathfrak{g}$ (not subalgebras!) such that

$$\mathfrak{g} = \bigoplus_{m=-\infty}^{\infty} g^m, \; (g^m, g^n) \subset g^{m+n} \quad \text{and} \quad g^m \neq 0 \quad \text{iff} \; -4 \leq m \leq 4.$$

Let us take a set of 9 generators of $\mathfrak{g}$, namely $e_1, f_1 \in \mathfrak{g}_1$ and $e_2, e_3, f_2, f_3, h_1, h_2, h_3 \in \mathfrak{g}_0$, such that

$$(e_i, f_j) = \delta_{ij} h_i, \quad (h_i, e_j) = \alpha_{ij} e_j, \quad (e_i, f_j) = -\alpha_{ij} f_j,$$

where

$$[\alpha_{ij}] := \begin{bmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}, \; \alpha \in \mathbf{R}.$$

Then the following commutators form bases of g^m

$$g^0 : h_1, h_2, h_3$$

$$\begin{array}{ll} g^1 : e_1, e_2, e_3 & g^{-1} : f_1, f_2, f_3 \\ g^2 : (e_1, e_2), (e_1, e_3) & g^{-3} : (f_1, f_2), (f_1, f_3) \\ g^3 : ((e_1, e_2), e_3) := e_{(3)} & g^{-2} : ((f_1, f_2), f_3) := f_{(3)} \\ g^4 : (((e_1, e_2), e_3), e_1) := e_{(4)} & g^{-4} : (((f_1, f_2), f_3), f_1) := f_{(4)} \end{array}$$

Recall that the series $D(2,1,\alpha)$ may be considered as a deformation of the orthosymplectic algebra $\mathrm{osp}(4,1) = D(2,1-1)$ whose even part splits $(\mathrm{osp}(4,1))_0 = so(4) \oplus sp(1)$. Therefore the unprecedent decomposition $so(4) = so(3) \oplus so(3)$ is involved in occurring of Lie superalgebras $D(2,1,\alpha)$ which form the only continuous family of simple Lie superalgebras. The decomposition $so(4) = so(3) \oplus so(3)$ is also involved in Donaldson's construction of exotic structures of $\mathbb{R}^4$, see [21]. That is why the Lie superalgebras $D(2,1,\alpha)$, $\alpha \neq -1$, here are called *exotic* ones.

Let us put

$$g^+ := \bigoplus_{m=0}^{4} g^m, \; g^- := \bigoplus_{m=-4}^{0} g^m$$

and divide the set of all BCH- and BCH-invertible subalgebras $f \subset g$ in eight parts assuming that $f^{\pm} := f \cap g^{\pm}$ satisfy the following conditions (L.s. = Lie superalgebra, L.a. = Lie algebra).

$$\begin{array}{ll} \alpha : f^- = 0 & \alpha' : f^+ = 0 \\ \beta : f^- \text{ is even Lie algebra} & \beta' : f^+ \text{ is even Lie algebra} \end{array}$$

$\gamma : f^-$ and f^+ are L.s.

$\delta : f^-$ is $\mathbb{Z}_2$-graded L.a., f^+ is L.s. $\quad \delta' : f^+$ is $\mathbb{Z}_2$-graded L.a., f^- is L.s.

$\epsilon : f^+$ and f^- are $\mathbb{Z}_2$-graded L.a.,

In [8] all BCH-invertible and, in particular, BCH- subalgebras of $D(2,1,\alpha)$ was listed. Below we show a part of BCH-invertible and BCH-subalgebras which belong to (ϵ) and are of types $(-1.3), (-2,3)$ and $(-3,3)$, i.e. the heighest (smallest) $\mathbb{Z}$-gradation of its elements is 3 $(-1,-2,-3$ resp.).

$$(-1,3)$$

$$\{f_1\}\oplus k^0\oplus\{e_1,\epsilon e_3\}\oplus\{(e_1,e_2)\}\oplus g^3, \epsilon=0,1,$$
$$k^0\subset\{2h_1-h_2,h_3\}, \left(k^0=0 \text{ or } \{2h_1-h_2+h_3\}\right)^*$$
$$\alpha\neq 0,\ \{f_1\}\oplus k^0\oplus\{\epsilon e_2,e_3\}\oplus\{(e_1,e_3)\}\oplus g^3,\ \epsilon=0,1,\ k^0\subset\left\{\frac{2h_1}{\alpha}-h_3,h_2\right\},$$
$$\left(k^0=0 \text{ or } \{2h_1-h_2+h_3\}\right)^*$$

$$(-2,3)$$

$$\{(f_1,f_2)\}\oplus\{f_1,\epsilon_1 f_2\}\oplus k^0\oplus\{\epsilon_2 e_2,e_3\}\oplus\{(e_1,e_3)\}\oplus g^3,\ \epsilon_{1,2}=0,1,\epsilon_1+\epsilon_2\neq 0,$$
$$(\epsilon_1=1,\ \epsilon_2=0)\Rightarrow\alpha\neq 0,\ k^0\subset\{2h_1-\alpha h_3,h_2\},\ \epsilon_1=\epsilon_2=1\Rightarrow k^0\supset\{h_2\}$$
$$*\{(f_1,f_2)\}\oplus\{f_1\}\oplus\mu^0\oplus\{e_3\}\oplus\{(e_1,e_3)\}\oplus g^3$$
$$\alpha\neq 0,\ \{(f_1,f_2)\}\oplus\{f_1\}\oplus k^0\oplus\{\epsilon e_2,e_3\}\oplus g^3,\ \epsilon=0,1,\ k^0\subset\left\{\frac{2h_1}{\alpha}-h_3,h_2\right\},$$
$$\left(k^0=0 \text{ or } \{2h_2-(\alpha+2)h_2-\alpha h_3\}\right)^*$$
$\alpha\neq 0,\ \{(f_1,f_2)\}\oplus\{\epsilon f_2\}\oplus k^0\oplus\{e_3\}\oplus\{(e_1,e_3)\}\oplus g^3,\ \epsilon$ and k^0 are as before

$\alpha\neq 0,\ \{(f_1,f_2)\}\oplus k^0\oplus\{e_3\}\oplus g^3,\ k^0$ is as before

$$\{(f_1,f_3)\}\oplus\{f_1,\epsilon f_3\}\oplus k^0\oplus\{e_2,\epsilon_2 e_3\}\oplus\{(e_1,e_2)\}\oplus g^3,\ \epsilon_{1,2}=0,1,\ \epsilon_1+\epsilon_2\neq 0,$$
$$k^0\subset\{2h_1-h_3,h_2\},\epsilon_1=\epsilon_2=1\Rightarrow k^0\supset\{h_3\}$$
$$\{(f_1,f_3)\}\oplus\{f_1\}\oplus\mu^0\oplus\{e_2\}\oplus\{(e_1,e_2)\}\oplus g^3,k^0\subset\{2h_2-h_3,h_2\},\ (k^0=\mu^0)^*$$
$$\{(f_1,f_3)\}\oplus\{f_1\}\oplus k^0\oplus\{e_2\epsilon e_3\}\oplus g^3,\ \epsilon=0,1,\ k^0\subset\{2h_1-h_3,h_2\},$$
$$\left(k^0=0 \text{ or } \{2h_1-h_2-(1+2\alpha)h_3\}\right)^*$$
$\{(f_1,f_3)\}\oplus\{\epsilon f_3\}\oplus k^0\oplus\{e_2\}\oplus\{(e_1,e_2)\}\oplus g^3,\ \epsilon$ and k^0 are as before

$\{(f_1,f_3)\}\oplus k^0\oplus\{e_2\}\oplus g^3, k$ is as before

Observe an asymmetry caused by the permutation $\{1,3,2\}$ of indices 1, 2,3.

(53) $$(-3,3)$$

$$g^{-3}\oplus k^0\oplus g^3,\ k^0\supset\{h_1-h_2-\alpha h_3\},\ \left(k^0=\{h_1-h_2-\alpha h_3\} \text{ or } \{h_1-h_2-\alpha h_3,h_2-h_3\}\right)^*$$
$$\alpha=0,\ m_{\epsilon,k^0}:=g^{-3}\oplus\{\epsilon(f_1,f_2)\}\oplus k^0\oplus\{(e_1,e_2)\}\oplus g^3,\ \epsilon=0,1,k^0\supset\{h_1-h_2\}$$

$\alpha = 0$, $g^{-3} \oplus \{\epsilon(f_1, f_2)\} \oplus k^0 \oplus \{e_3\} \oplus g^3$, ϵ and k^0 are as before,
$\alpha = 0$, $n_{\epsilon,k^0} := g^{-3} \oplus \{(f_1, f_2)\} \oplus \{\epsilon f_3\} \oplus k^0 \oplus \{e_3\} \oplus \{(e_1, e_2)\} \oplus g^3$, $\epsilon = 0, 1$,

(54)
$$k^0 \supset \begin{cases} \{h_1 - h_2\}, & \epsilon = 0 \\ \{h_1 - h_2, h_3\}, & \epsilon = 1. \end{cases}$$

The remaining algebras belonging to the set (ϵ) may be obtained by a reflection in which e_i is replaced by f_i and g^i by g^{-i}.

Comments. (c) Recall that for $\alpha = 0$ the corresponding Lie superalgebra g is not simple and semi-simple because then its subspace I given by

(55)
$$I \oplus \{f_3\} \oplus \{h_3\} \oplus \{e_3\} = g$$

becomes an unsolvable ideal. The BCH-subalgebra $m_{1,\{h_1,h_2\}}$, see (53), the weak BCH subalgebra $n_{1,\{h_1-h_2,h_3\}}$, see (54), and the ideal I fulfil a simple relation, namely

$$m_{1,\{h_1-h_2\}} = n_{1,\{h_1-h_2,h_3\}} \cap I.$$

(cc) The considered spectrum of BCH- and BCH-invertible subalgebras of $D(2,1,\alpha)$ remains unaltered in the sense of a set of subspaces of the corresponding 17-dimensional space if α changes in the set $\mathbf{R} - \langle -1, 0 \rangle$. This is not a case of a spectrum of all subalgebras of $D(2,1,\alpha)$ because the subspace

$$\{h_1\} \oplus \{e_2, e_3\} \oplus \{(e_1, e_2 + \beta e_3)\} \oplus g^3 \oplus g^4, \ \beta \neq 0$$

becomes a subalgebra when $\alpha = 1$. Thus the computed spectra are more regular with respect to α than spaces of all subalgebras of $D(2,1,\alpha)$.

(ccc) Note that the above method may be insufficient in order to determine all BCH- and BCH-invertible subalgebras of non-contragradient Lie superalgebras like $P(n)$ and $Q(m)$.

5.3. Subalgebras of the super-Poincaré algebra

This algebra admits decompositions $g = g_1 \oplus g_0, g_0 = g_0^t \oplus g_0^r$ in the sense of vector spaces, where $g_1 = \{Q_1, Q_2, Q_3, Q_4\}$ consists of generators of supersymmetry transformations, $g_0^t = \{P_1, P_2, P_3, P_4\}$ corresponds to space-time and $g_0^r = \{J_1, \ldots, J_6\}$

is the Lorentz algebra. Recall non-zero commutation relations in the traditional notation with $\{\cdot,\cdot\}$ for anti-commutators

$$\begin{array}{llll}
\{Q_1,Q_3\}=P_1, & \{Q_1,Q_4\}=P_2, & \{Q_2,Q_3\}=P_3, & \{Q_2,Q_4\}=P_4,\\
[J_1,Q_2]=-Q_1, & [J_2,Q_1]=\frac{1}{2}Q_1, & [J_2,Q_2]=-\frac{1}{2}Q_2, & [J_3,Q_1]=Q_2,\\
[J_4,Q_4]=Q_3, & [J_5,Q_3]=-\frac{1}{2}Q_3, & [J_5,Q_4]=\frac{1}{2}Q_4, & [J_6,Q_3]=-Q_4,\\
[J_1,P_3]=-P_1, & [J_1,P_4]=-P_2, & [J_2,P_1]=\frac{1}{2}P_1, & [J_2,P_2]=\frac{1}{2}P_2,\\
[J_2,P_3]=-\frac{1}{2}P_3, & [J_2,P_4]=-\frac{1}{2}P_4, & [J_3,P_1]=P_3, & [J_3,P_2]=P_4,\\
[J_4,P_2]=P_1, & [J_4,P_4]=P_3, & [J_5,P_1]=-\frac{1}{2}P_1, & [J_5,P_2]=\frac{1}{2}P_2,\\
[J_5,P_3]=-\frac{1}{2}P_3, & [J_5,P_4]=\frac{1}{2}P_4, & [J_6,P_1]=-P_2, & [J_6,P_3]=-P_4,\\
[J_1,J_2]=-J_1, & [J_1,J_3]=-2J_2, & [J_2,J_3]=-J_3, & \\
[J_4,J_5]=J_4, & [J_4,J_6]=2J_5, & [J_5,J_6]=J_6, &
\end{array}$$

Thus commutation relations for the distinguished parts of g look as follows:

$$\{g_1,g_1\}=g_0^t,\ [g_1,g_0^r]=g_1,\ [g_0^r,g_0^t]=g_0^t,[g_0^r,g_0^r]=g_0^r,$$
$$[g_1,g_0^t]=[g_0^t,g_0^t]=0.$$

For any subalgebra $p=p_1\oplus p_0^t\oplus p_0^r\subset g$, $(p_i^*=g_i^*\cap p)$, the conditions (31) of being a BCH- and (34) of being a BCH-invertible Lie superalgebra are resp.

$$[p_1,p_0^r]=0,$$
$$[\{p_1,p\},p_r^0]=0.$$

These equalities allow us to describe all BCH- and weak BCH subalgebras of g. The first ones are automorphic images of

$$\{Q_1,Q_3\}\oplus g_0^t\oplus\{J_1,J_4\},\ \{Q_1,Q_2,Q_3\}\oplus g_0^t\oplus\{J_4\},\ g_1\oplus g_0^t$$

(here and below $\{\cdot\}$ denotes a linear envelope). The weak BCH subalgebras of the super-Poincaré algebra g are g-automorphic to

$$\{Q_1,Q_3\}\oplus g_0^t\oplus\{J_1,J_4,J_2+J_5\}.$$

6. SUPERBUNDLES FORMED BY MEANS OF THE BCH-FORMULA

The main two points in which Lie supergroups from Sec. 4 differ from ordinary Lie groups are the absence of an explicitely defined global structure of such supergroups and failing of group axioms by their group-like operation: particularly non-associativity in some domains. Both these discrepancies cause some obstructions when we try to construct some superbundles relevant to said supergroups nevertheless they do not make such constructions impossible.

Observe that additional relations should be assumed in order to define cocycles taking values in a supergroup and an equivalence relation between them. That is so because if M is a base space with a covering (U_α), $\mathcal{G}$ is a Lie supergroup admitting a Lie superalgebra g and $(c_{\alpha\beta}(x))$ are $\mathcal{G}$-cocycles of functions $c_{\alpha\beta} : U_{\alpha\beta} \to \mathcal{G}$, $U_{\alpha\beta} := U_\alpha \cap U_\beta$, then sets of admissible values of $c_{\alpha\beta}(x)$ may depend on x, α, β.

Therefore a C^∞- or C^ω-fibre bundle $\mathcal{P} \to M$ having as fibres Lie supergroups $\mathcal{P}_x$ (in any sense of Sec. 4) will be used as a domain of regarded cocycles. We call $\mathcal{P}_x$ *structural Lie supergroups.* Let p_x be Lie superalgebras of P_x. We assume that p_x are subalgebras of a Lie superalgebra g called a *structural Lie superalgebra.* Below we describe separately three cases when the subalgebras p_x are BCH-, weak BCH and non-BCH-invertible Lie superalgebras.

I. *The subalgebras p_x are BCH-Lie superalgebras.* Assume that $\mathcal{G}$ is a Jacobi-Lie or a generalized Jacobi-Lie supergroup. In this case the definition of a $\mathcal{P}$-cocycle $(c_{\alpha\beta})$ is as in geometry of fibre bundles, namely

$$(56) \qquad c_{\alpha\beta} * c_{\beta\gamma} = c_{\alpha\gamma} \text{ on } U_{\alpha\beta\gamma}, \quad c_{\alpha\beta} * c_{\beta\alpha} = \text{id on } U_{\alpha\beta}, \quad c_{\alpha\beta}(x) \in \mathcal{P}_x.$$

The definition of equivalent $\mathcal{P}$-cocycles is also standard

$$(57) \quad (c_{\alpha\beta}) \sim (c'_{\alpha\beta}) \Leftrightarrow (c'_{\alpha\beta}) = s_\alpha * c_{\alpha\beta} * s_\beta^{-1} \text{ on } U_{\alpha\beta},\ s_\alpha(x) \in \mathcal{P}_x \text{ for } x \in U_\alpha.$$

If $\mathcal{P}$ is of C^∞-class and $\mathcal{P}_x$ are isomorphic to a Lie group P, whose a maximal compact subgroup is K, then $(c_{\alpha\beta})$ is C^∞-equivalent to a cocycle of a principal K-bundle over M in consequence of the Steenrod theorem, cf. [20], [7], [11]. Note that K is a subgroup of Lie group P_0, whose Lie algebra is isomorphic to $g_0 \cap p_x$, where g is a structural Lie superalgebra.

Let us make a remark that if p_x are BCH-Abelian then sets of local sections determining higher cohomologies of Čech are well defined be means of standard cohomological formulae for sections of $\mathcal{P}$.

II. *The subalgebras p_x are weak BCH.* Let $\mathcal{G}$ be a standard Lie supergroup or one of its extensions. In the present case $\mathcal{P}$-cocycles are also defined by means of formula (56); however, the system of equations determined in the present case by this formula is different from such a system in the previous case. This is so because the present system includes equations coming from associativity of superposition of transition functions on intersections of more then three covering sets. In the case of intersections of four sets we have

$$c_{\alpha\beta} * (c_{\beta\gamma} * c_{\gamma\delta}) = (c_{\alpha\beta} * c_{\beta\gamma}) * c_{\gamma\delta} \quad \text{on} \quad U_{\alpha\beta\gamma\delta}. \tag{58}$$

Let us observe that the algebraic formula of systems of equations for $c_{\alpha\beta}$ is the same in both cases as well as for group valued cocycles if

$$U_{\alpha\beta\gamma\delta} = \emptyset \quad \text{for all } \alpha, \beta, \gamma, \delta. \tag{59}$$

In order to define equivalent cocycles in this point let us introduce a subbundle $\mathcal{Q} \subset \mathcal{P}$ whose fibres $\mathcal{Q}_x$ are Lie supergroups having Lie superalgebras q_x and then make of $\mathcal{Q}$ a domain for local sections s_α. The fibres $\mathcal{Q}_x$ will be called *groups of local isomorphisms.* The relation of equivalence obtained in this way is

$$(c_{\alpha\beta}) \sim (c'_{\alpha\beta}) \Leftrightarrow c'_{\alpha\beta} = (s_\alpha * c_{\alpha\beta}) * s_\beta^{-1}, \quad c_{\alpha\beta}(x) \in \mathcal{P}_x,\ x \in U_{\alpha\beta}, \\ s_\alpha(x) \in \mathcal{Q}_x,\ x \in U_\alpha. \tag{60}$$

The resulting sets of «transition functions» $(c'_{\alpha\beta})$ for all sections s_α are cocycles if p_x are BCH-Lie superalgebras and for all $s, s_1 \in \mathcal{Q}_x$ and $g, g_1 \in \mathcal{P}_x$ it is

$$\begin{aligned} &(g * s^{-1}) * (s * g_1) = g * g_1, \qquad (s * g) * (g^{-1} * s_1) = s * s_1, \\ &(s * g) * g_1 = s * (g * g_1), \qquad (g * s) * s_1 = g * (s * s_1). \end{aligned} \tag{61}$$

The above equations follow from more simple ones

$$(s * g) * g_1 = s * (g * g_1), (g * s) * g_1 = g * (s * g_1), (g * g_1) * s = g * (g_1 * s) \tag{62}$$

being a consequence of the following equalities

$$(p_{x,1}, (p_x, q_{x,1})) = (p_{x,1}, (q_x, p_{x,1})) = (q_{x,1}, (p_x, p_{x,1})). \tag{63}$$

If p_x, q_x are block-triangular matrices with parameters r, s, t and r, s', t' resp., see (52), then the above system of equations is equivalent to

$$2r - \max(2s + s', s + t + s', 2s + t) \geq 0 \tag{64}$$

and the conditions of BCH-invertibility are

$$2r - t - 2s \geq 0, \quad 2r - 3t < 0. \tag{65}$$

If p_x are isomorphic to the super-Poincaré algebra then relations (61) imply that $q_x = 0$.

The equivalence classes of $\mathcal{P}$-cocycles defined by (60) will be called $(\mathcal{P} - \mathcal{Q})$*-cocycles.* We point out classes of equivalent fibre bundles which correspond to $(\mathcal{P} - \mathcal{Q})$-cocycles in a similar way as classes of principal bundles correspond to classes of group valued cocycles.

Let $(c_{\alpha\beta})$ be a $\mathcal{P}$-cocycle representing a $(\mathcal{P} - \mathcal{Q})$-cocycle $[c_{\alpha\beta}]$ and $\pi_\alpha : B^{(\alpha)} \to U_{\underline{\alpha}_1}$ be fibre bundles of the same class as $\mathcal{G}$ and $\mathcal{P}$ and such that $\pi_\alpha^{-1}(x)$ are subsets of $\mathcal{P}_x$ and the following compatibility conditions hold

$$\mathcal{B}_x^{(\alpha)} := \{c_{\alpha\beta}(x) * a_x^{(\beta)} \mid a_x^{(\beta)} \in \mathcal{B}_x^{(\beta)}, \quad x \in U_{\alpha\beta}\}. \tag{66}$$

The desired fibre bundle may be defined as a quotient of disjoint sum of $\mathcal{B}_x^{(\alpha)}$ by the following relation

$$a_x^{(\alpha)} \sim a_x^{(\beta)} \Leftrightarrow a_x^{(\alpha)} = c_{\alpha\beta}(x) * a_x^{(\beta)}, \text{ where } a_x^{(\alpha)} \in \mathcal{B}_x^{(\alpha)}, a_x^{(\beta)} \in \mathcal{B}_x^{(\beta)}. \tag{67}$$

The resulting fibre bundles may be understood as *superbundles.*

One can quickly prove that equivalent $\mathcal{P}$-cocycles determine equivalent fibre bundles in usual sense if the family of bundles $\mathcal{B}^\alpha$ is fixed. This is a consequence of (60) and (61).

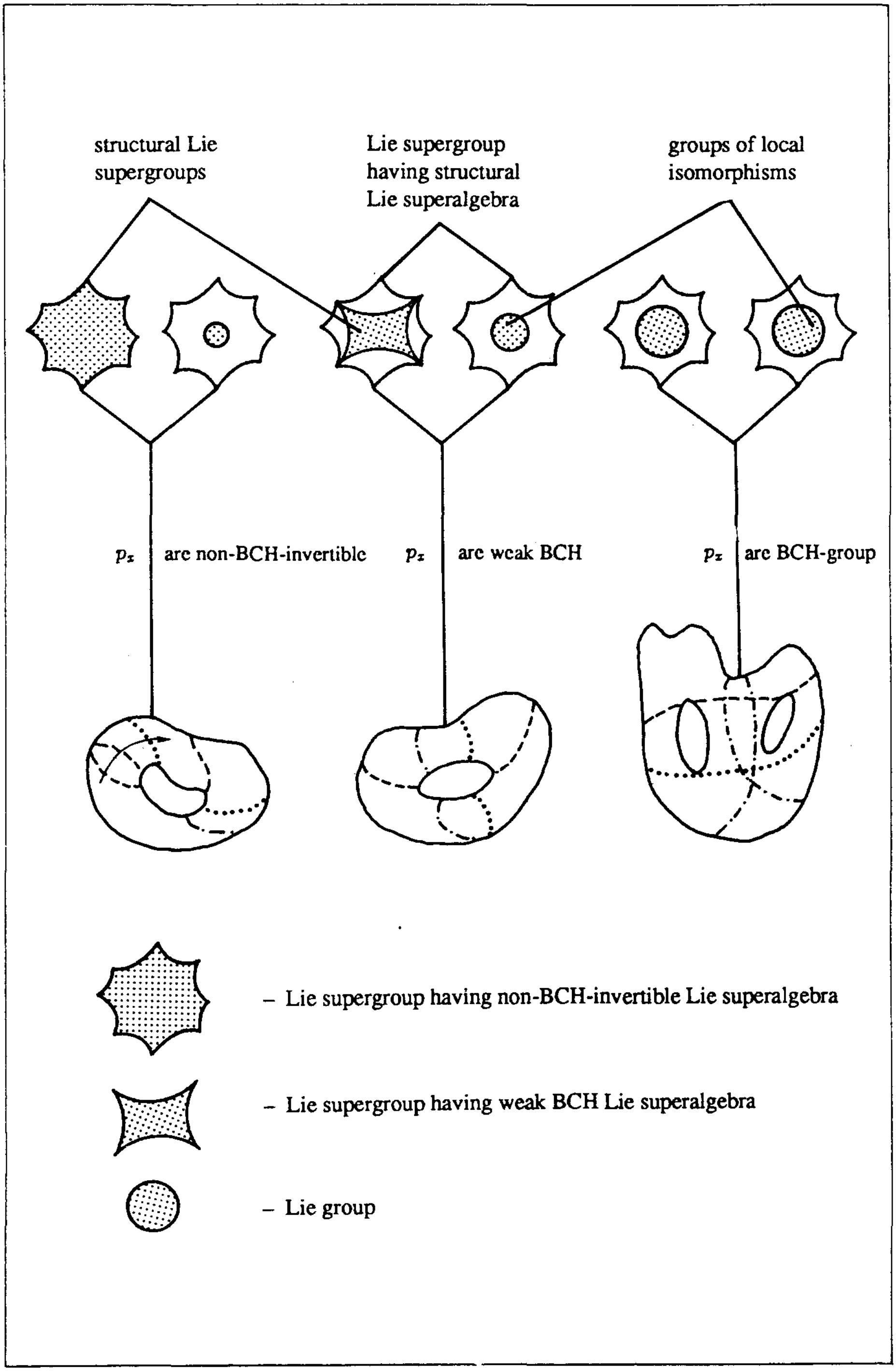
Let us also note that if fibre bundles $\mathcal{P}$ and $\mathcal{B}^{(\alpha)}$ are locally trivial then the obtained superbundles are locally trivial too.

A passing to the inductive limit in the set of $(\mathcal{P} - \mathcal{Q})$-cocycles over M related to all admissible coverings in done in the same way as it is in the case of group valued cocycles. The resulting objects may be considered as some first Čech cohomologies in certain local groupoids. Let us recall that the non-associativity of $\mathcal{P}_x$ may imply equations of «constraints» like (58) if we change a covering.

III. *The subalgebras* p_x *are not BCH-invertible.* In this case only a part of relations (56) and (60) may be considered simultaneously. We distinguish them by means of an order $\prec$ in the set I of indices α of covering sets U_α. Let us assume that

(O1) $\alpha \prec \beta \quad \Leftrightarrow \quad (\alpha \neq \beta \text{ and } U_{\alpha\beta} \neq \emptyset)$,
(O2) $\alpha \prec \beta \quad \Rightarrow \quad (\beta \prec \alpha \text{ is not true})$,
(O3) $(\alpha \prec \beta, \ \beta \prec \gamma \text{ and } U_{\alpha\beta} \neq \emptyset) \Rightarrow \alpha \prec \gamma$.

Such an order may be established by means of a linear order $<$ in I if we put $\alpha \prec \beta \Leftrightarrow (\alpha < \beta$ and $U_{\alpha\beta} \neq 0)$.


structural Lie supergroups
Lie supergroup having structural Lie superalgebra
groups of local isomorphisms
p_x are non-BCH-invertible
p_x are weak BCH
p_x are BCH-group
– Lie supergroup having non-BCH-invertible Lie superalgebra
– Lie supergroup having weak BCH Lie superalgebra
– Lie group

In the present case relations of cocycles and of their equivalences are given by (56) with additional condition $\alpha \prec \beta \prec \gamma$ and by (60) with $\alpha \prec \beta$ and exchange of s_β^{1} by ${}^{r}s_\beta^{-1}$ where ${}^{r}s_\beta^{-1} * s_\beta = \mathrm{id}$ on U_β.

The formulae (61-64) do not require others changes and their implications remain unaltered. However (65) must be replaced with $2r - t - 2s < 0$ since p_x are not BCH-invertible.

In this case superbundles and their equivalency may be defined by means of (66-67) together with $\alpha \prec \beta$.

The corresponding local Čech cohomologies irrelevant to particular coverings also require dealing with the orders « $\prec$ » while passing to the inductive limit.

The relations between structural groups, groups of local isomorphisms and base spaces satisfying (59) in non-group cases are illustrated by the above figure.

7. WHAT IS A FIRST SUPER CHERN CLASS?

Assume that g admits a decomposition into direct sum of vector spaces

$$g = a \oplus j, \quad \dim a = k \tag{68}$$

in which a is a BCH-Abelian subalgebra of g and j is an ideal of g.

Then the projection

$$\pi : g \overset{\text{on}}{\to} j$$

is a homomorphism.

Recall that there are many semi-simple Lie superalgebras admitting decomposition (68). This is a case when a is odd-commutative, i.e. $a = a_1$ and its bracket is 0-form, j is semi-simple and $(j, a) = 0$ [13, 13]. Another example of this decomposition occurs if we consider $D(2,1,0)$ and put $a = \{e_3\}$ or $\{f_3\}$, $j = I$, where I is the ideal defined in (55).

Let $\mathcal{G}$ be a standard Lie supergroup corresponding to the Lie superalgebra g which hold (68) and let $\mathcal{A} \subset \mathcal{G}$ be a local abelian group which is the a-part of $\mathcal{G}$. Then the map $\pi_e : \mathcal{G} \to \mathcal{A}$ given by

$$\pi_e(e^x) := e^{\pi(x)}$$

is a homomorphism by virtue of (44). For any manifold M the homomorphism π_e determines the map Π_e of the set of $\mathcal{G}$-cocycles on M at the set of $\mathcal{A}$-cocycles on M. In a similar way one defines maps Π_{id} between sets of a $\mathcal{G}$-cocycles and sets of

$\mathcal{A}$-cocycles when $\mathcal{G}$ is a modified or a toroidal Lie supergroup corresponding to g and $A \subset \mathcal{G}$ is the a-part of $\mathcal{G}$ (for definitions of the said supergroups see points d and e in Sec. 4).

If $\mathcal{G}$ is a modified Lie supergroup and M is a paracompact C^∞-manifold then each $\mathcal{A}$-cocycle is C^∞-equivalent to 0-cocycle. In order to have non-zero C^∞-classes of $\mathcal{A}$-cocycles assume the following decomposition

$$a = a^1 \oplus a^2, \quad (a^1, j) = 0, \quad \dim a^{1,2} = k_{1,2}, \ k_1 > 0$$

and consider a toroidal Lie supergroup $\mathcal{G}$, whose a-part is given by

$$A \cong \mathbf{R}^{k_1}/_{L_{k_1}} \oplus \mathbf{R}^{k_2},$$

where L_{k_1} denotes a lattice of a k_1-torus. Then there is a simple 1-1 correspondence between $\mathcal{A}$-cocycles and $\frac{U(1) \times \ldots \times U(1)}{k_1}$-cocycles. Furthermore, if $\mathcal{G}$-cocycles are equivalent in the sense of any definition from the previous section then their images by Π_{id} are equivalent $\mathcal{A}$-cocycles.

Let us observe that in the case where $g = u(n), j = su(n)$, L_1 is given by the natural period of exponent and $\mathcal{G}$-cocycles correspond to $U(n)$-bundles we can obtain the first Chern class of $U(n)$-bundles in this way.

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